# Steady and Oscillatory Subsonic and Supersonic Aerodynamics around Complex Configurations

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**A general formulation for steady and oscillatory, subsonic and supersonic, potential linearized aerodynamic flows around complex configurations is presented. A linear integral equation relating the unknown potential on the surface of the body to the known downwash is used. The formulation is applied to the analysis of flowfields around wings and wing-body combinations. The surface is divided into small quadrilateral elements which are approximated with hyperboloidal surfaces. The potential is assumed to be constant within each element. This yields a set of linear algebraic equations. The coefficients are evaluated analytically. Numerical results for steady and oscillatory, subsonic and supersonic flows indicate that the method, is not only more general and flexible than other available methods, but is also fast, accurate, and in excellent agreement with existing results.**

## Nomenclature

$\mathbf{a}_i$ = Eqs. (18) and (19)
$a_\infty$ = freestream speed of sound
$b$ = span
$b_{hk}$ = Eqs. (9) and (30)
$C_p$ = pressure coefficient, $2(p-p_\infty)/\rho_\infty U_\infty{}^2$
$C_l$ = lift distribution coefficient, $C_{pl}-C_{pu}$
$C_{l\alpha}$ = $(\partial C_l/\partial\alpha)_{\alpha=0}$
$c$ = chord
$c_{hk}$ = Eqs. (8) and (29)
$E$ = Eq. (4)
$H$ = Eq. (26)
$k$ = reduced frequency, $\omega l/U_\infty$
$l$ = reference length
$M$ = Mach number, $U_\infty/a_\infty$
$\mathbf{n}$ = Eq. (20)
$\mathbf{N}$ = outward normal to surface $\Sigma$
$NX, NY$ = number of elements in $X, Y$ directions
$\mathbf{P}$ = point of the surface $\Sigma$
$\mathbf{P}_*$ = control points, $(X_*, Y_*, Z_*)$
$\mathbf{R}$ = Eq. (17)
$R$ = Eq. (3)
$t$ = time
$T$ = $a_\infty \beta t/l$
$x, y, z$ = space coordinates
$X, Y, Z$ = nondimensional Prandtl-Glauert coordinates, Eq. (2)
$U_\infty$ = freestream velocity
$\alpha$ = angle of attack
$\beta$ = $|1-M^2|^{1/2}$
$\Sigma$ = surface surrounding body and wake
$\Sigma_k$ = surface element
$\Sigma_A$ = surface of aircraft
$\Sigma_W$ = surface of wake
$\tau$ = thickness ratio
$\Phi$ = velocity potential, $U_\infty x+\varphi$
$\varphi$ = perturbation velocity potential
$\phi$ = $\varphi/U_\infty l$
$\hat{\phi}$ = Eqs. (A1) and (A13)
$\phi_k$ = values of $\phi$, at centroid of element $\Sigma_k$
$\omega$ = frequency of oscillation
$\Omega$ = compressible reduced frequency, $\omega l/a_\infty\beta = kM/\beta$
$AR$ = aspect ratio
$TR$ = taper ratio
$\Delta\phi$ = $\phi_u-\phi_l$
$\nabla$ = gradient in $X, Y, Z$ space
$\odot$ = superproduct, Eq. (22)
$\|\mathbf{a}\|$ = supernorm of a vector, Eq. (23)

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## I. Introduction

THE structural design of aircraft requires the evaluation of the aerodynamic loads. For flexible structures, these loads must be evaluated by an iterative process, since the loads depend upon the geometry of the surface of the aircraft, while the geometry depends upon the aerodynamic loads. In addition, the structural optimal design is subject to aeroelastic constraints such as divergence and flutter. These constraints require accurate evaluations of steady and oscillatory aerodynamic loads. Again, the optimal design requires an iterative process, since the resizing of the structure changes the stability boundary. Similar considerations can be made about the evaluation of aerodynamic derivatives and gust response. Needless to say, these iterative processes are best performed on high speed digital computers.

The automation of these iterative schemes requires that the mathematical modeling of the problem must be general, flexible, and efficient. Computational methods for structures are currently quite well developed, and it can be stated that these methods satisfy the requirements of generality, flexibility and efficiency. Computational methods for the evaluation of steady and unsteady, subsonic and supersonic aerodynamic loads around complex configurations have been recently developed. These methods are based upon the assumption of potential aerodynamics, which is largely adequate for this purpose, and this is the problem considered here: computational formulations for steady and oscillatory, subsonic and supersonic potential aerodynamic flows around complex configurations (excluded are the effects of nonlinearities, in particular, transonic and hypersonic flows). In these conditions, the problem can be reduced to the solution of a surface integral equation relating the unknown (usually the strength of source, doublet or vortex distribution on the surfaces of the aircraft and the wake) to the downwash, which is known from the boundary conditions on the surface of the aircraft. Only these methods are considered here.

An excellent review of the state-of-the-art of aerodynamics for complex configurations is given in Ref. 1. Only a short summary is included here. The first work on the flowfield around three-dimensional bodies is probably the one by Hess and Smith.[2] Their method uses constant-strength source-elements to solve the problem of steady subsonic flow around nonlifting bodies. The intensity of the source-element is obtained by imposing the tangency boundary conditions. This method has been extended to lifting bodies by Rubbert and Saaris[3,4] by including doublet and vortex panels. Similar methods are used by Hess[5] and Labrujere, Loeve, and Slooff.[6] A different approach is used by Woodward,[7] who uses lifting surface elements for the representation of the body. For oscillatory subsonic aerodynamics, extensions of the doublet-lattice method by Albano and Rodden[8] are obtained by Rodden, Giesing, and Kalman,[9] and Giesing, Kalman, and Rodden,[10] either by placing unsteady lifting surface elements on the surface of the body or by using the method of images combined with the slender body theory. In the supersonic range, complex configurations are analyzed in the works of Carmichael and Woodward,[11] Woodward,[7] Baals, Robin, and Harris,[12] for steady flow. Finally, Appa and Smith[13] have proposed a formulation for unsteady supersonic flow around complex configurations, but numerical results have not been obtained yet.

It may be noted that all the available computational methods around complex configurations, while general, are usually quite cumbersome to use and always require human intervention to define the suitable type of element (source, doublet, vortex, etc.) to be used. Thus, the use of these methods is to be considered as an art, and therefore it cannot be easily automated. In other words, none of the methods currently available satisfies all the requirements of generality, flexibility, and efficiency necessary for automated design. Furthermore, for oscillatory flows around complex configurations, only techniques based upon the doublet-lattice method[8] exist in the subsonic range (see Refs. 9 and 10), while no method is available in the supersonic one.[1] A method that eliminates these problems is presented here.

## II. Outline of the Method

The method presented here is based upon a new formulation recently developed by Morino.[14,15] The formulation is based upon the Green function method applied to the equation of the velocity potential, which yields a representation of the potential $\varphi$, at any point, $\mathbf{P}_*$, in the flowfield (control point), in terms of the values of the potential and its normal derivative on the surface, $\Sigma$, surrounding the body and the wake. The integral equation is obtained by imposing that the value of the potential at $\mathbf{P}_*$ approaches the value of $\varphi$ on the surface, as $\mathbf{P}_*$ approaches a point on the surface. It may be noted that this integral equation is different from the ones currently used in aerodynamics in that it does not impose that the boundary condition on the down-wash be satisfied, but rather makes use of the continuity of the potential as the control point approaches the surface, $\Sigma$. The tangency boundary conditions are automatically satisfied by the type of representation obtained with the Green theorem. A feasibility analysis by Morino and Kuo[16] indicates that the method is fast and accurate, at least for finite-thickness wings in steady and oscillatory flows. Results for subsonic and supersonic flows around complex configurations are given in Ref. 17. The main results presented in Ref. 17 are summarized in this paper. More detailed expositions are available in Refs. 18–21. The surface of the aircraft and its wake is divided into small quadrilateral elements. Each element is replaced by a hyperboloidal surface defined by the four corner points of the element. In this process, the continuity of the surface is maintained, although discontinuities in the slopes are introduced. The unknown is assumed to be constant within each element and therefore, the integral equation reduces to a system of algebraic equations. The numerical formulations for subsonic and supersonic flows are presented in Secs. III and IV. For the sake of clarity and conciseness, only the steady state formulations are discussed in detail. The unsteady cases are outlined in the Appendix. Numerical results for steady and oscillatory, subsonic and supersonic flows are presented in Sec. V and indicate that the method is not only intrinsically more general and flexible than other existing ones, but is also efficient, fast, and accurate.

## III. Steady Subsonic Flow

The Green theorem for the linearized equation of the steady subsonic aerodynamic potential flow may be written as

$$4\pi E(\mathbf{P}_*)\phi(\mathbf{P}_*) = -\oiint_{\Sigma_A} \frac{\partial \phi}{\partial N}\frac{1}{R}\, d\Sigma_A + \oiint_{\Sigma_A} \phi \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_A + \iint_{\Sigma_W} \Delta\phi \frac{\partial}{\partial N_u}\left(\frac{1}{R}\right) d\Sigma_W \quad (1)$$

where

$$X = x/\beta l \quad Y = y/l \quad Z = z/l \quad \phi = \varphi/U_\infty l \quad (2)$$

($\beta = |1-M^2|^{1/2}$ and $l$ is a characteristic length), and $\Sigma_A$ and $\Sigma_W$ are the surfaces of the aircraft and the wake, in the $X, Y, Z$ space, while $\mathbf{N}$ is the outward normal to $\Sigma_A$ and $\mathbf{N}_u$ is the upper normal to $\Sigma_W$. In addition,

$$R = [(X-X_*)^2+(Y-Y_*)^2+(Z-Z_*)^2]^{1/2} \quad (3)$$

where $X, Y, Z$ are the dummay variables of integration, and

$$\begin{aligned} E(\mathbf{P}_*) &= 1 \quad \text{outside} \quad \Sigma_A \\ &= \tfrac{1}{2} \quad \text{on} \quad \Sigma_A \\ &= 0 \quad \text{inside} \quad \Sigma_A \end{aligned} \quad (4)$$

On the surface $\Sigma_A$, Eq. (1) is an integral equation relating the potential $\phi$ to its normal derivative, which is given by the tangency boundary condition

$$\partial\phi/\partial N = -N_x/\beta \quad (5)$$

where $N_x$ is the component of $\mathbf{N}$ along the $x$-axis (direction of the unperturbed flow). In Eq. (5), higher order terms have been neglected. This is consistent with the process of linearization of the differential equation for $\varphi$.[14] It may be worth noting that while the differential equation and boundary conditions have been linearized, the location of the surface of the body is the correct one, since no linearization of the surface boundary location is feasible for complex configurations. Note that $\Delta\varphi$ is constant along a streamline and equal to the value at the trailing edge

$$\Delta\varphi = \Delta\varphi_{TE} \quad (6)$$

since no pressure difference can exist on the wake.

In order to obtain an approximate solution for the integral equation, the surfaces $\Sigma_A$ and $\Sigma_W$ are divided into small quadrilateral elements, $\Sigma_k$. The values of $\phi$ and $\partial\phi/\partial N$ are assumed to be constant within each element. The collocation method is then used, that is, Eq. (1) is satisfied at the centroids, $\mathbf{P}_h$, of the element, $\Sigma_h$. This yields

$$[\delta_{hk} - c_{hk} - w_{hk}]\{\phi_k\} = [b_{hk}]\left\{\left(\frac{\partial\phi}{\partial N}\right)_k\right\} \quad (7)$$

where $\delta_{hk}$ is the Kronecker delta,

$$c_{hk} = \left[\frac{1}{2\pi}\iint_{\Sigma_k} \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_k\right]_{\mathbf{P}_* = \mathbf{P}_h} \quad (8)$$

$$b_{hk} = \left[-\frac{1}{2\pi}\iint_{\Sigma_k} \frac{1}{R}\, d\Sigma_k\right]_{\mathbf{P}_* = \mathbf{P}_h} \quad (9)$$

In addition, $w_{hk} = 0$ for the elements not in contact with the trailing edge, while for the elements in contact with the trailing edge,

$$w_{hk} = \left[\pm\frac{1}{2\pi}\iint_{\Sigma_k'} \frac{\partial}{\partial N_u}\left(\frac{1}{R}\right) d\Sigma_k'\right]_{\mathbf{P}_* = \mathbf{P}_h} \quad (10)$$

where $\Sigma_k'$ is the strip of the wake (bounded by two streamlines) emanating from the element $\Sigma_k$. The upper (lower) sign must be used for the upper (lower) side of the wake. In deriving Eq. (7), the value of $\phi_{TE}$ is approximated with the value of $\phi$ at the centroid of the elements adjacent to the trailing edge. This is reasonable in view of the Kutta condition.[16]

The surface elements $\Sigma_k$ are approximated by a hyperbolic paraboloid (hyperboloidal element, Fig. 1)

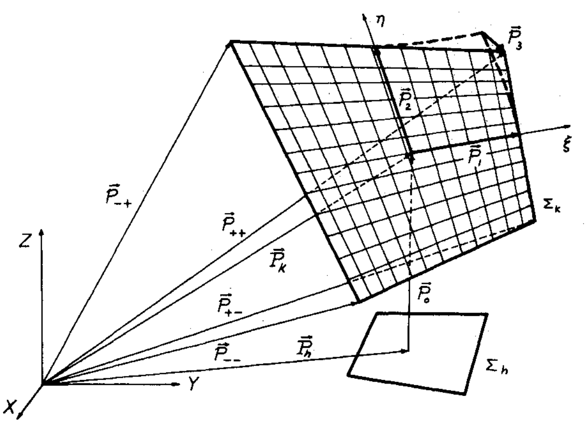


**Fig. 1 Hyperboloidal element.**

$$\mathbf{P} = \mathbf{p}_0 + \xi\mathbf{p}_1 + \eta\mathbf{p}_2 + \xi\eta\mathbf{p}_3 \tag{11}$$

where $\mathbf{p}_i$ are obtained in terms of the locations of the four corner points as

$$\begin{Bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{Bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathbf{p}_{++} \\ \mathbf{p}_{+-} \\ \mathbf{p}_{-+} \\ \mathbf{p}_{--} \end{Bmatrix} \tag{12}$$

Note that quadrilateral hyperboloidal elements can be combined to yield a closed surface. Using these elements, the coefficients $c_{hk}$ and $b_{hk}$ can be evaluated analytically,[18] as

$$c_{hk} = I_D(1,1) - I_D(1,-1) - I_D(-1,1) + I_D(-1,-1) \tag{13}$$

$$b_{hk} = I_S(1,1) - I_S(1,-1) - I_S(-1,1) + I_S(-1,-1) \tag{14}$$

with [using $-\pi/2 \leq \tan_p^{-1}(\;) \leq \pi/2$]

$$I_D(\xi,\eta) = (1/2\pi)\tan_p^{-1}(\mathbf{R}\times\mathbf{a}_1\cdot\mathbf{R}\times\mathbf{a}_2/|\mathbf{R}|\,\mathbf{R}\cdot\mathbf{a}_1\times\mathbf{a}_2) \tag{15}$$

and (neglecting $\partial\mathbf{n}/\partial\xi$ and $\partial\mathbf{n}/\partial\eta$)

$$I_S(\xi,\eta) = -\frac{1}{2\pi}\left\{-\mathbf{R}\times\mathbf{a}_1\cdot\mathbf{n}\frac{1}{|\mathbf{a}_1|}\sinh^{-1}\left(\frac{\mathbf{R}\cdot\mathbf{a}_1}{|\mathbf{R}\times\mathbf{a}_1|}\right) + \mathbf{R}\times\mathbf{a}_2\cdot\mathbf{n}\frac{1}{|\mathbf{a}_2|}\sinh^{-1}\left(\frac{\mathbf{R}\cdot\mathbf{a}_2}{|\mathbf{R}\times\mathbf{a}_2|}\right) + \mathbf{R}\cdot\mathbf{n}\tan_p^{-1}\left(\frac{\mathbf{R}\times\mathbf{a}_1\cdot\mathbf{R}\times\mathbf{a}_2}{|\mathbf{R}|\,\mathbf{R}\cdot\mathbf{a}_1\times\mathbf{a}_2}\right)\right\} \tag{16}$$

where

$$\mathbf{R}(\xi,\eta) = \mathbf{P} - \mathbf{P}_h = \mathbf{p}_0 + \xi\mathbf{p}_1 + \eta\mathbf{p}_2 + \xi\eta\mathbf{p}_3 - \mathbf{P}_h \tag{17}$$

$$\mathbf{a}_1(\xi,\eta) = \partial\mathbf{R}/\partial\xi = \mathbf{p}_1 + \eta\mathbf{p}_3 \tag{18}$$

$$\mathbf{a}_2(\xi,\eta) = \partial\mathbf{R}/\partial\eta = \mathbf{p}_2 + \xi\mathbf{p}_3 \tag{19}$$

$$\mathbf{n}(\xi,\eta) = \mathbf{a}_1\times\mathbf{a}_2/|\mathbf{a}_1\times\mathbf{a}_2| \tag{20}$$

Similar expressions are used for $w_{hk}$.[18]

Equation (7) can be solved numerically to yield the values of the unknowns, $\phi_k$. In all the results presented here, the pressure is obtained from the linearized Bernoulli theorem

$$C_p = -(2/\beta)\,\partial\phi/\partial X \tag{21}$$

(with the derivative evaluated by finite difference), while the wake is assumed to be composed of straight vortex-lines, emanating from the trailing edge.

## IV. Steady Supersonic Flow

In order to extend the subsonic formulation to the supersonic case, it was convenient to develop a new vector algebra, called "supersonic vector algebra" or, in short, "super-algebra,"[19] In this algebra, a new product between vectors, "supersonic dot product" or, in short, "super-product," is introduced:

$$\mathbf{a}\odot\mathbf{b} = a_X b_X - a_Y b_Y - a_Z b_Z \tag{22}$$

(read "a super b"). Note that $\mathbf{a}\odot\mathbf{a}$ is not necessarily positive. Therefore, the "super-norm" of a vector $\mathbf{a}$ is defined as

$$\|\mathbf{a}\| = |\mathbf{a}\odot\mathbf{a}|^{1/2} \tag{23}$$

while $|\mathbf{a}|$ indicates the regular norm

$$|\mathbf{a}| = (\mathbf{a}\odot\mathbf{a})^{1/2} \tag{24}$$

With these notations, the Green theorem for steady supersonic flow can be written as[19]

$$2\pi E(\mathbf{P}_*)\phi(\mathbf{P}_*) = -\oiint_{\Sigma_A}\frac{\partial\phi}{\partial N^c}\frac{H}{\|\mathbf{R}\|}d\Sigma_A + \oiint_{\Sigma_A}\phi\frac{\partial}{\partial N^c}\left(\frac{H}{\|\mathbf{R}\|}\right)d\Sigma_A + \oiint_{\Sigma_W}\Delta\phi\frac{\partial}{\partial N_u{}^c}\left(\frac{H}{\|\mathbf{R}\|}\right)d\Sigma_W \tag{25}$$

where

$$\begin{aligned} H &= 1 \quad \text{for} \quad X_* - X > [(Y_* - Y)^2 + (Z_* - Z)^2]^{1/2} \\ H &= 0 \quad \text{for} \quad X_* - X \leq [(Y_* - Y)^2 + (Z_* - Z)^2]^{1/2} \end{aligned} \tag{26}$$

while the conormal derivative, $\partial/\partial N^c$, is given by

$$\partial/\partial N^c \equiv -N_X\,\partial/\partial X + N_Y\,\partial/\partial Y + N_Z\,\partial/\partial Z = -\mathbf{N}\odot\boldsymbol{\nabla} \tag{27}$$

For supersonic leading edges, each side of the configuration is considered independently. Diaphragms can be used to obtain the same advantage for subsonic leading edges.

Using the same numerical procedure described for the subsonic flow yields

$$[\delta_{hk} - c_{hk} - w_{hk}]\{\phi_k\} = [b_{hk}]\{(\partial\phi/\partial N^c)_k\} \tag{28}$$

where

$$c_{hk} = \left[+\frac{1}{\pi}\iint_{\Sigma_k}\frac{\partial}{\partial N^c}\left(\frac{H}{\|\mathbf{R}\|}\right)d\Sigma\right]_{\mathbf{P}_* = \mathbf{P}_h} \tag{29}$$

$$b_{hk} = \left[-\frac{1}{\pi}\iint_{\Sigma_k}\frac{H}{\|\mathbf{R}\|}d\Sigma\right]_{\mathbf{P}_* = \mathbf{P}_h} \tag{30}$$

and the definition of $w_{hk}$ is similar to the one for subsonic flow. For hyperboloidal elements completely inside the Mach forecone, $c_{hk}$ and $b_{hk}$ are still given by Eqs. (13) and (14), with Eqs. (15) and (16) replaced by

$$I_D(\xi,\eta) = -\frac{1}{\pi}\tan_p^{-1}\left(\frac{\mathbf{R}\times\mathbf{a}_1\odot\mathbf{R}\times\mathbf{a}_2}{\|\mathbf{R}\|\,\mathbf{R}\cdot\mathbf{a}_1\times\mathbf{a}_2}\right) \tag{31}$$

and

$$I_S(\xi,\eta) = -\frac{1}{\pi}\frac{1}{\|\mathbf{n}\|^2}\left\{-\mathbf{R}\times\mathbf{a}_1\odot\mathbf{n}F_1(\xi,\eta) + \mathbf{R}\times\mathbf{a}_2\odot\mathbf{n}F_2(\xi,\eta) + \mathbf{R}\cdot\mathbf{n}\tan_p^{-1}\left(\frac{\mathbf{R}\times\mathbf{a}_1\odot\mathbf{R}\times\mathbf{a}_2}{\|\mathbf{R}\|\,\mathbf{R}\cdot\mathbf{a}_1\times\mathbf{a}_2}\right)\right\} \tag{32}$$

where $\mathbf{R}$, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{n}$ are given by Eqs. (17–20), while

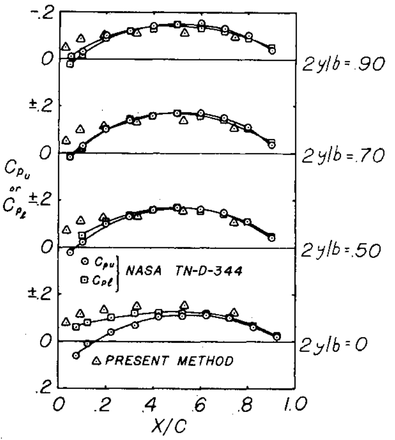


**Fig. 2 Pressure coefficient for circular biconvex symmetric rectangular wing with $AR = 3$, $\tau = 0.05$, $\alpha = 0°$ (thickness problem), $M = 0.24$, $NX = NY = 7$, and comparison with results of Ref. 22.**

$$F_j(\xi,\eta) = \frac{\text{sign}(\mathbf{R}\odot\mathbf{a}_j)}{\|\mathbf{a}_j\|}\sinh^{-1}\left(\frac{\|\mathbf{R}\|\,\|\mathbf{a}_j\|}{\|\mathbf{R}\times\mathbf{a}_j\|}\right) \qquad (\mathbf{a}_j\odot\mathbf{a}_j>0)$$
$$= \frac{\|\mathbf{R}\|}{\mathbf{R}\odot\mathbf{a}_j} \qquad (\mathbf{a}_j\odot\mathbf{a}_j=0) \tag{33}$$
$$= -\frac{1}{\|\mathbf{a}_j\|}\sin^{-1}\left(\frac{\mathbf{R}\odot\mathbf{a}_j}{\|\mathbf{R}\times\mathbf{a}_j\|}\right) \qquad (\mathbf{a}_j\odot\mathbf{a}_j<0)$$

If the element is partially inside the Mach forecone, special expressions for $c_{hk}$ and $b_{hk}$ must be used.[19] If the element is completely outside the Mach forecone, then $c_{hk} = b_{hk} = 0$.

Finally, neglecting higher order terms, the boundary condition yields

$$\partial\phi/\partial N^c = -N_x/\beta \tag{34}$$

In addition, for all the results presented here, the wake is never considered (supersonic trailing edges), while the pressure is obtained from Eq. (21) with the derivative evaluated by finite difference. Furthermore, all the results presented here were obtained by using a diaphragm to separate the upper and lower sides of the aircraft. For the elements on the diaphragm, both $\phi$ and $\partial\phi/\partial N^c$ are unknown, while two different equations are obtained by writing Eq. (28) for the upper and lower side, respectively. The solution of the problem was obtained from the system derived by writing Eq. (28) for the upper and lower sides of the body and the diaphragm.

## V. Numerical Results

The formulation presented here was imbedded into a computer program, SOSSA ACTS (Steady and Oscillatory, Subsonic and Supersonic Aerodynamics for Aerospace Complex Transportation Systems). Typical results obtained with this program are presented in this section. It may be noted that numerical results indicate[20,21] that the effect of the angle-of-attack $\alpha$ is important only in the boundary conditions but not in the definition of the geometry of the surface. Therefore using $\alpha = 0$ in the geometry, the problem becomes linear in $\alpha$, and thus the thickness and the lift problem can be decoupled. In all the results presented here, symmetry (thickness problem) or antisymmetry (lift problem) with respect to the $z$-axis has been used. Wings in steady subsonic flow are considered in Figs. 2 and 3, which present a comparison with the experimental results of Lessing, Troutman, and Menees.[22] The results are relative to a rectangular wing with aspect ratio $AR = 3$. The airfoil consists of a biconvex circular-arc section, 5% thick, with sharp leading and trailing edges. Figure 2 shows the thickness effect, that is the pressure distribution on upper and lower sides of the wing

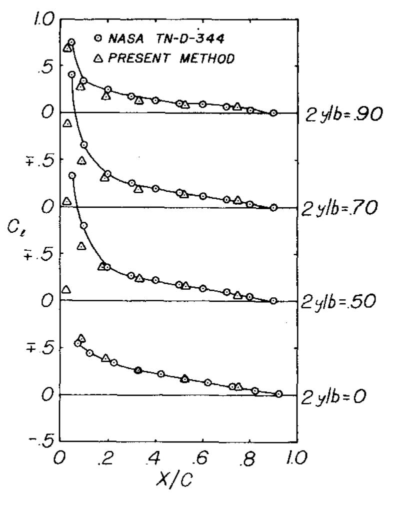


**Fig. 3 Lift coefficient for circular biconvex symmetric rectangular wing with $AR = 3$, $\tau = 0.05$, $\alpha = 5°$ (lift problem), $M = 0.24$, $NX = NY = 7$, and comparison with results of Ref. 22.**

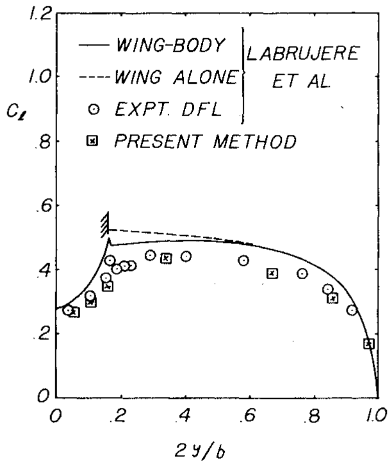


**Fig. 4 Lift coefficient for wing-body configuration with $\alpha_w = 6°$, $\alpha_B = 0°$, $M = 0$, total number of elements $N = 200$, and comparison with results of Ref. 6.**

at zero angle of attack and $M = 0.24$. Figure 3 presents the lift distribution for $\alpha = 5°$ and $M = 0.24$. These results were obtained by using 196 elements on the whole wing (i.e., $NX = NY = 7$ where $NX$ and $NY$ are the number of elements on the upper right side of the wing in $X$ and $Y$ directions, respectively). Comparison with various lifting surface theories is given in Ref. 17. These results are similar to the ones given in Ref. 16, and therefore, are not given here.

Results for wing-body configurations in steady subsonic flow are compared in Fig. 4, with the results presented by Labrujere, Loeve, and Slooff.[6] The results were obtained for $M = 0$ and a rectangular midpositioned wing with chord $c = 1$, span $b = 6$, thickness ratio $\tau = 0.9$, and $\alpha_w = 6°$. The body is at zero angle of attack and is composed of a forebody with length $L_A = 2$ and

$$r = \tfrac{1}{2} - \tfrac{1}{8}(x - x_{LE})^2 \tag{35}$$

a midsection of length $L_M = 1$ and radius, $r = 0.5$, and an aftbody of constant radius $r = 0.5$ and length $L_A = 9$. No wake is used for the body. The number of elements is 200 on the whole configuration ($NX = 5$, $NY = 4$ on the wing, $NX = 2$, $NY = 3$ on the forebody, $NX = 5$, $NY = 3$ on the midsection, and $NX = NY = 3$ on the aftbody).

Results for subsonic oscillatory flow are presented in Fig. 5, for the same wing considered in Fig. 2, oscillating in bending mode

$$z = 0.18043\,|2y/b| + 1.70255\,|2y/b|^2 - 1.13688\,|2y/b|^3 + 0.25387\,|2y/b|^4 \tag{36}$$

with $k = \omega c/2U_\infty = 0.47$, and $M = 0.24$. Figure 5 presents a comparison with the results of Lessing et al.[22] The results are obtained with $NX = NY = 7$ on the wing. On the wake $NY = 7$, while $NX = 10$ (the size in the $x$-direction for the elements of the wake is equal to the ones of the wing).

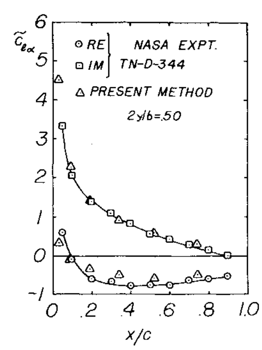


**Fig. 5 Pressure coefficient for rectangular wing oscillating in bending mode with $k = \omega c/2U_\infty = 0.47$, $M = 0.24$, $AR = 3$, $NX = NY = 7$, and comparison with results of Ref. 22.**

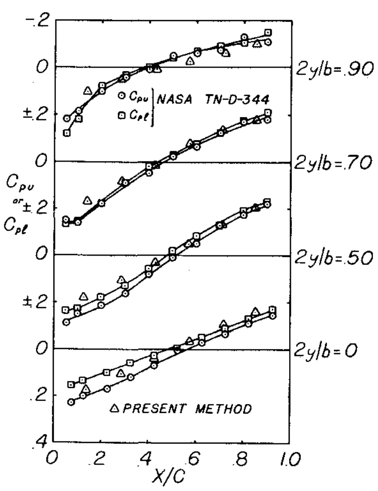


**Fig. 6 Pressure coefficient for a circular biconvex symmetric rectangular wing with $AR = 3$, $\tau = 0.05$, $\alpha = 0°$ (thickness problem), $M = 1.3$, $NX = NY = 7$, and comparison with results of Ref. 22.**

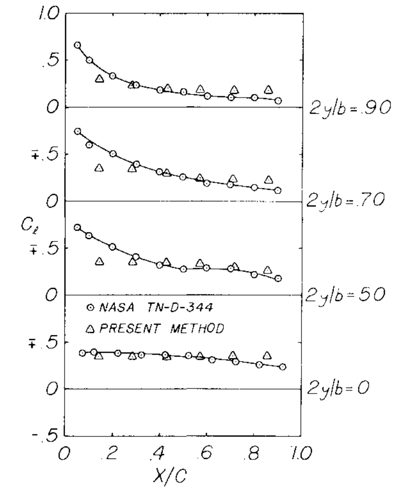


**Fig. 7 Lift coefficient for a circular biconvex symmetric rectangular wing with $AR = 3$, $\tau = 0.05$, $\alpha = 5°$ (lift problem), $M = 1.3$, $NX = NY = 7$, and comparison with results of Ref. 22.**

Results for supersonic flow are presented in Figs. 6–10. Figure 6 shows the distribution of the pressure coefficient $C_p$ on the lower and upper surfaces for the same wing considered in Fig. 2, with $\alpha = 0°$ and $M = 1.3$, and Fig. 7 presents the lift coefficient for the same wing for $\alpha = 5°$ and $M = 1.3$. The results, obtained with $NX = NY = 7$, are compared with the ones obtained by Lessing, et al.[22] Figure 8 presents the results for a delta wing with supersonic leading edges and

$$m \equiv \beta/\tan \Lambda = 1.2 \tag{37}$$

The results obtained with $NX = NY = 15$ are compared with the exact solution (see, for instance, Jones and Cohen[23]). A wing-body combination in supersonic flow with $\alpha_w = 1.92$, $\alpha_B = 0$ and $M = 1.48$ is considered in Fig. 9. The number of elements is 556 on the whole configuration ($NX = NY = 10$ on the wing, $NX = NY = 3$ on the forebody, $NX = 10$, $NY = 3$ on the middle section). The results are compared with the ones by Nielsen[24] and Woodward, Tinoco, and Larsen[25] (see also Ref. 1). Finally, results for oscillatory supersonic flow are presented in Fig. 10 for the same problem considered in Fig. 5. The results are obtained for $k = \omega c/2U_\infty = 0.1$, $M = 1.3$, and $NX = NY = 10$ and are compared with the ones by Lessing, et al.[22]

## VI. Conclusions

A new method for analyzing linearized steady and oscillatory, subsonic and supersonic potential aerodynamics around complex configurations has been presented. Numerical results have been obtained for wing-body configurations in steady flows, and for finite-thickness wings in oscillatory flows. It should be noted that the results presented here are believed to be the first ones ever obtained for supersonic oscillatory flows around non-zero-thickness configurations.

The main advantages of the present method with respect to existing ones are its generality as well as its flexibility and simplicity for use in automated design. It is apparent that this method is more general than any existing one. It may be worth noting that the general formulation presented in Refs. 14 and 15 includes general time-dependent motion. Furthermore, the method is very flexible and simple to use. The use of quadrilateral hyperboloidal elements defined in terms of their corner points, is one of the original features of the method.‡ Other original features of the method are the simplicity of the expression for the coefficients and the commonality between subsonic and supersonic, steady and oscillatory flows. These make the method extremely efficient from a practical point of view. Finally, the method is accurate and fast, despite the fact that these are very preliminary results and thus no effort has been made yet in order to minimize the computational time: as an example, the computer time for the results for subsonic wing-body configurations (200 elements) presented in Fig. 4, is 775 sec on the IBM 360/50 of Boston University's Computing Center. For the supersonic wing-body configurations (Fig. 9, 556 elements) the time is 1715 sec.

The results are usually in excellent agreement with existing ones, although in a few cases, an improvement in accuracy appears to be desirable. This is especially true for the oscillatory flows for which the evaluation of the coefficients is not as accurate as for the steady flows. This is probably due to the assumption that $e^{-i\Omega R}$ and

$$e^{-i\Omega R}(1+i\Omega R) = 1+\tfrac{1}{2}\Omega^2R^2+O(\Omega^3R^3) \tag{38}$$

can be approximated by its values at the centroids of the elements [see Eqs. (A8) and (A9)].§ This assumption can be used only for small values of $\Omega$. An analysis of convergence is given in Ref. 16; a more complete analysis of convergence is given in Ref. 26. It should be noted that the preliminary

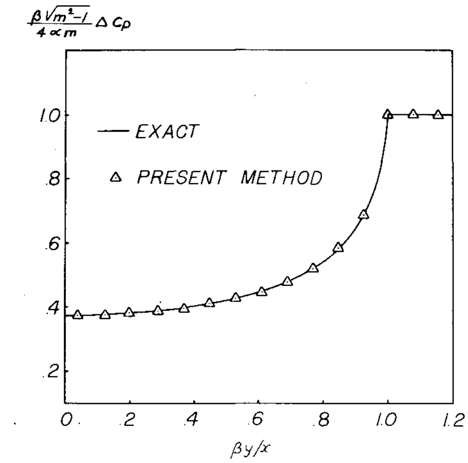


**Fig. 8 Pressure coefficient for delta wing with $m = \beta/\tan \Lambda = 1.2$, $NX = NY = 15$, and comparison with exact results.**

‡ It may be noted that the hyperboloidal elements are not introduced for the purpose of increasing the accuracy, but only to reduce the number of elements necessary to describe complex configurations and thereby increase the flexibility of the program.

§ Similar considerations are valid for supersonic flow, see Eqs. (A15) and (A16).

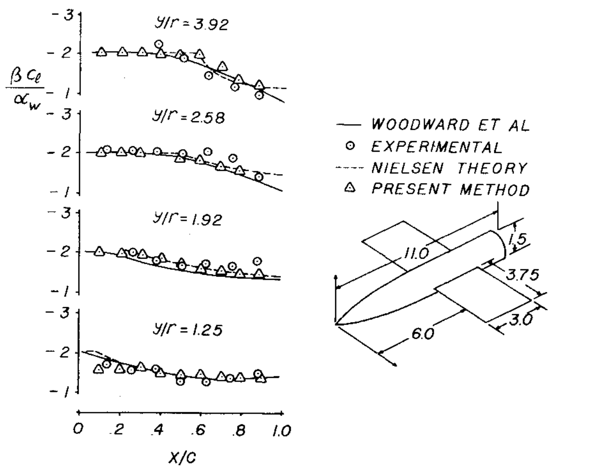


**Fig. 9 Lift coefficient for wing-body configuration with $\alpha_w = 1.92°$, $\alpha_B = 0°$, $M = 1.48$, total number of elements $N = 556$, and comparison with results of Refs. 24 and 25.**

results presented here are intended to validate the new integral formulation introduced in Refs. 14 and 15; the numerical formulation is still very crude and a more refined formulation is necessary in order to achieve an increase in efficiency and accuracy. This is especially needed for unsteady state with $\Omega \gg 1$, in particular if the Mach number is closed to one ($M = 1$ yields $\Omega = \infty$). Such a formulation is now underway and employs the finite-element technique.[27] The unknowns are assumed to be the values of the potential at the corner points of the elements. A hyperboloidal distribution for the potential is assumed within the element (first-order finite-element representation). It may be worth noting that the formulation presented here might be regarded as a zeroth-order finite-element representation. Another point under investigation is the use of finite-difference for the evaluation of $\partial\varphi/\partial X$ in the pressure coefficient. A formulation using the analytical derivative of Eq. 1 for the evaluation of $\partial\varphi/\partial X$ is considered in Ref. 28. Also under consideration is the question of the diaphragm for supersonic flow: recent results indicate that, for subsonic-leading edge configurations, the same results are obtained with or without the diaphragm.[26]

In conclusion, the results presented here indicate that the method can be applied to steady and oscillatory, subsonic and supersonic flows around complex configurations and is not only more general and flexible than existing ones (Refs. 2–12) but also fast and accurate.

## Appendix—Oscillatory Flow

In this Appendix, the formulation of the main body of this paper is extended to the oscillatory flow due to small oscillations of an aircraft traveling at subsonic or supersonic speed. For the subsonic flow, consider the complex potential $\hat{\phi}$ such that

$$\phi(X, Y, Z, T) = \hat{\phi}(X, Y, Z)\, e^{i\Omega(T+MX)} \qquad \text{(A1)}$$

with

$$T = a_\infty \beta t/l \qquad \Omega = \omega l/a_\infty \beta = KM/\beta \qquad \text{(A2)}$$

The Green theorem for $\hat{\phi}$ is

$$4\pi E(\mathbf{P}_*)\hat{\phi}(\mathbf{P}_*) = -\oiint_\Sigma \frac{\partial\hat{\phi}}{\partial N}\frac{e^{-i\Omega R}}{R}\, d\Sigma + \oiint_\Sigma \hat{\phi}\frac{\partial}{\partial N}\left(\frac{e^{-i\Omega R}}{R}\right) d\Sigma \qquad \text{(A3)}$$

where $\Sigma$ surrounds body and wake. Using the same procedure used for the steady state, one obtains

$$[\delta_{hk} - \hat{c}_{hk} - \hat{w}_{hk}]\{\hat{\phi}_k\} = [\hat{b}_{hk}]\{(\partial\hat{\phi}/\partial N)_k\} \qquad \text{(A4)}$$

where

$$\hat{c}_{hk} = \left[\frac{1}{2\pi}\iint_{\Sigma_k} \frac{\partial}{\partial N}\left(\frac{e^{-i\Omega R}}{R}\right) d\Sigma_k\right]_{\mathbf{P}_* = \mathbf{P}_h} \qquad \text{(A5)}$$

$$\hat{b}_{hk} = \left[-\frac{1}{2\pi}\iint_{\Sigma_k} \frac{e^{-i\Omega R}}{R}\, d\Sigma_k\right]_{\mathbf{P}_* = \mathbf{P}_h} \qquad \text{(A6)}$$

while $\hat{w}_{hk} = 0$ for the elements not in contact with the trailing edge and

$$\hat{w}_{hk} = \left[\pm\frac{1}{2\pi}\iint_{\Sigma_k'} e^{ik(X_k - X)/\beta}\frac{\partial}{\partial N}\left(\frac{e^{-i\Omega R}}{R}\right) d\Sigma_k'\right]_{\mathbf{P}_* = \mathbf{P}_h} \qquad \text{(A7)}$$

for the elements in contact with the trailing edge. In deriving Eq. (A7) the condition that no pressure difference exists across the wake has been used (see Eq. A10). The coefficients $\hat{c}_{hk}$ and $\hat{b}_{hk}$ may be approximated as

$$\hat{c}_{hk} = \left\{\frac{1}{2\pi}[e^{-i\Omega R}(1 + i\Omega R)]_{\mathbf{P} = \mathbf{P}_k} \iint_{\Sigma_k} \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_k\right\}_{\mathbf{P}_* = \mathbf{P}_h} \qquad \text{(A8)}$$

$$\hat{b}_{hk} = \left\{\frac{-1}{2\pi}[e^{-i\Omega R}]_{\mathbf{P} = \mathbf{P}_k} \iint_{\Sigma_k} \frac{1}{R}\, d\Sigma_k\right\}_{\mathbf{P}_* = \mathbf{P}_h} \qquad \text{(A9)}$$

For hyperboloidal elements, the integrals are evaluated with the same expressions used for the steady case. Finally, the coefficients $\hat{w}_{hk}$ are evaluated by dividing the wake-strip into small elements and using expressions similar to Eq. (A8). This was found to be more efficient than the analytical integration in the $x$ direction, previously used.[16] The pressure coefficient is obtained from the linearized Bernoulli theorem in the form[16]

$$\tilde{C}_p = C_p e^{-i\Omega T} = -(2/\beta)\, e^{-ik\beta X}\, \partial(\hat{\phi}\, e^{ikX/\beta})/\partial X \qquad \text{(A10)}$$

with the derivative evaluated by finite-difference. Finally, the boundary conditions, neglecting higher order terms, are given by

$$\partial\hat{\phi}/\partial N = N_Z(ik\tilde{Z} + (1/\beta)\partial\tilde{Z}/\partial X)\, e^{-i\Omega MX} \qquad \text{(A11)}$$

where $\tilde{Z}(X, Y)$ is the mode of vibration.

Similarly for supersonic flow, the Green theorem is given by[14,19]

$$2\pi E(\mathbf{P}_*)\hat{\phi}(\mathbf{P}_*) = -\oiint_\Sigma \frac{\partial\hat{\phi}}{\partial N^c}\frac{H}{R'}\cos\Omega R'\, d\Sigma + \oiint_\Sigma \hat{\phi}\frac{\partial}{\partial N^c}\left(\frac{H}{R'}\cos\Omega R'\right) d\Sigma \qquad \text{(A12)}$$

where $\hat{\phi}$ is such that

$$\phi(X, Y, Z, T) = \hat{\phi}(X, Y, Z)\, e^{i\Omega(T - MX)} \qquad \text{(A13)}$$

and $R' = \|\mathbf{R}\|$. Using the same method used for the steady case, one obtains

$$[\delta_{hk} - \hat{c}_{hk} - \hat{w}_{hk}]\{\hat{\phi}_k\} = [\hat{b}_{hk}]\{(\partial\hat{\phi}/\partial N^c)_k\} \qquad \text{(A14)}$$

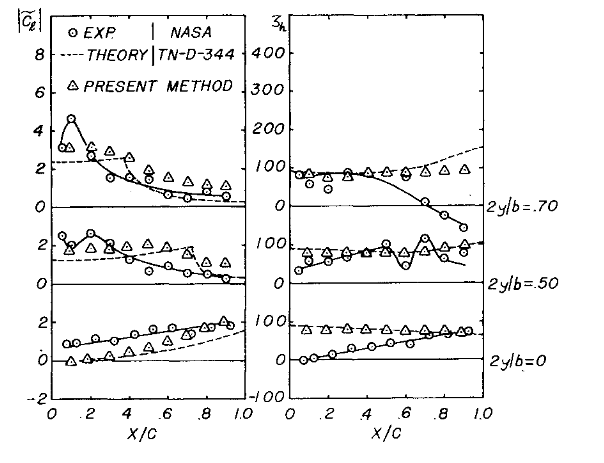


**Fig. 10 Absolute value $|\tilde{c}_l|$ and phase angle $\zeta_n$ of lift coefficient for rectangular wing oscillating in bending mode with $k = \omega c/2U_\infty = 0.1$, $M = 1.3$, $AR = 3$, $NX = NY = 10$, and comparison with results of Ref. 22.**

The coefficients are evaluated as

$$\hat{c}_{hk} = \left\{ \frac{1}{\pi} [\cos \Omega R' + \Omega R' \sin \Omega R']_{P = P_k} \times \iint_{\Sigma_k} \frac{\partial}{\partial N^c} \left( \frac{H}{R'} \right) d\Sigma_k \right\}_{P_* = P_h} \quad (A15)$$

$$\hat{b}_{hk} = \left\{ -\frac{1}{\pi} [\cos \Omega R']_{P = P_k} \iint_{\Sigma_k} \frac{H}{R'} d\Sigma_k \right\}_{P_* = P_h} \quad (A16)$$

where the integrals are evaluated with the same expressions used for the steady case. For the case of supersonic trailing edge considered here, $\hat{w}_{hk} = 0$; a procedure similar to the one described for oscillatory subsonic flow, can be used for other cases. Finally, the pressure is evaluated from

$$\tilde{C}_p = -(2/\beta) e^{-i\beta KX} \partial(\hat{\phi} e^{-ikX/\beta})/\partial X \quad (A17)$$

(with the derivative evaluated by finite-difference) while the boundary conditions, neglecting higher order terms, are given by

$$\partial\hat{\phi}/\partial N^c = N_Z(ik\tilde{Z} + (1/\beta)\partial\tilde{Z}/\partial X) e^{i\Omega MX} \quad (A18)$$

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